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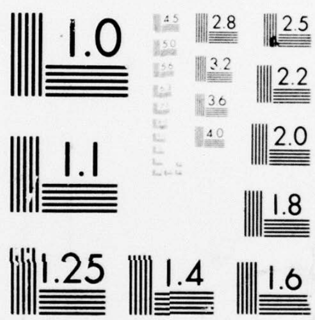

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APPROXIMATION OF A COMPLETELY MONOTONE FUNCTION

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ABSTRACT

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A function f on $(0,\infty)$ is completely monotone if it possesses derivatives of all orders, and the successive derivatives alternate in sign. It is shown that for each x the value of f(x) lies between any two consecutive partial sums of the expansion of f(x) in Taylor series. The given result can be applied to various functions such as the hypergeometric and confluent hypergeometric functions, which are widely used in applied mathematics. Some statistical applications are also given.

Key words and phrases: Hypergeometric; Confluent Hypergeometric; Exponential function; Gamma distribution.

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1. Introduction and main results. A function f on $(0,\infty)$ is said to be completely monotone if it possesses derivatives $f^{(n)}$ of all orders and

$$(-1)^n f^{(n)}(x) \geq 0, \; x > 0.$$

Typical examples of a completely monotone function are e^{-x} and $(1+x)^{-m}$, where m is a positive number. A wide class of functions which arise in applied mathematics have the completely monotone property. Consider, for example, the confluent hypergeometric and the hypergeometric functions, given by

$$\Phi(a,b;x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \cdot \frac{x^r}{r!}$$

$$\psi(a,b;c;x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \cdot \frac{x^r}{r!}$$

Where $(a)_r = a(a+1)\ldots(a+r-1)$. By the integral representation formulas (see e.g. Abramowitz and Stegun [1], 13.2.1, 15.3.1) we have that

$$\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)} \Phi(a,b;x) = \int_0^1 e^{xt} t^{a-1}(1-t)^{b-a-1} \, dt \qquad (1.1)$$

$$b > a > 0$$

$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \psi(a,b;c;x) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} \, dt \qquad (1.2)$$

$$c > b > 0$$

From (1.1) and (1.2) it is seen that the confluent hypergeometric function $\Phi(a,b;-x)$ is completely monotone for $b > a > 0$ and the hypergeometric function $\psi(a,b;c;-x)$ is completely monotone for $c > b > 0$, $a > 0$. If $a = b$ then $\Phi(a,b;-x) = e^{-x}$.

The completely monotone property is important in the theory of probability and statistical analysis. It is known that a function f on $(0,\infty)$ is the Laplace transform of a probability distribution if and only if it is completely monotone and $f(0) = 1$. From Feller ([2], X111.4, Criterion 1 and 2) we have that if f and g are completely monotone then the product fg is also monotone and that if f is completely monotone and g is a positive function with completely monotone derivative then f(g) is completely monotone (in particular, e^{-g} is completely monotone).

Let

$$S_n(x) = \sum_{r=0}^{n} f^{(r)}(0)\,\frac{x^r}{r!}$$

denote the partial sum of the Taylor series expansion of f(x). The following theorem shows that if f is completely monotone then for each x the value of f(x) lies between any two consecutive partial sums.

<u>Theorem 1.1</u>. Let n be a positive integer. If f is completely monotone then for each $x > 0$

$$S_{2n-1}(x) \le f(x) \le S_{2n-2}(x). \qquad (1.3)$$

Proof: Let

$$h(x) = f(x) - S_{2n-1}(x).$$

We have $h^{(r)}(0) = 0$ for $r = 0, 1, \ldots, 2n-1$. Since $h^{(2n-1)}(0) = 0$ and $h^{(2n)}(x) = f^{(2n)}(x) \ge 0$ therefore $h^{(2n-1)}(x) \ge 0$. Repeating

the argument we have that $h^{(2n-2)}(x) \geq 0$. Successive repetition of the argument yields $h(x) \geq 0$, establishing the first inequality in (1.3). The second inequality in (1.3) is proved similarly. □

The result of Theorem 1.1 for $f(x) = e^{-x}$ is known. The following theorem and its corollary extend that result. Theorem 1.3 below gives a monotonicity property of the tail of the exponential series. The proof of the theorem is omitted. Let

$$M_t(x) = e^{-x} - \sum_{r=0}^{t} (-x)^r/r!, \quad x \geq 0.$$

Theorem 1.1. If t is even (odd) then for all $x \geq 0$ and $0 \leq \nu \leq 1$

$$M_t(x) + (-1)^t \frac{x^{\nu+t}}{\Gamma(\nu+t+1)} \geq (\leq) 0. \tag{1.4}$$

Proof: Let $x \geq 0$, $0 \leq \nu \leq 1$ and let $L(\nu)$ denote the quantity on the left side of the inequality (1.4). Since $M_t(x) \leq (\geq) 0$ if t is even (odd) by Theorem 1.1, we have that $L(\nu) \geq (\leq) 0$ if t is even (odd) for $\nu = 0$ and 1.

Let $Z = x^{\nu+t}/\Gamma(\nu+t+1)$. We have

$$\begin{aligned} \partial^2 \log z/\partial\nu^2 &= -\partial\psi(\nu+t+1)/\partial\nu \\ &< 0 \end{aligned}$$

where ψ denotes the digamma function. Hence Z is either decreasing or increasing or first increasing then decreasing as ν varies from 0 to 1. Suppose that t is even. Then $L(\nu)$ is either decreasing or increasing or first increasing then decreasing as ν varies from 0 to 1. Since $L(\nu) \geq 0$ for $\nu = 0$

and 1, it follows that $L(\nu) \geq 0$ for $0 \leq \nu \leq 1$. Similarly, $L(\nu) \leq 0$ for $0 \leq \nu < 1$ if t is odd.

Corollary 1.1. If $0 \leq \nu \leq 1$ then for all $x \geq 0$

$$|M_t(x)| \leq x^{\nu+t}/\Gamma(\nu+t+1).$$

Let

$$m_{t+\nu}(x) = |M_t(x)|/x^{t+\nu}.$$

Theorem 1.3. As x varies from 0 to ∞, $m_{t+1}(x)$ increases and $m_{t+\nu}(x)$ first increases then decreases for $0 < \nu < 1$.

Application: Let

$$g_\nu(x) = \frac{x^{\nu-1}e^{-x}}{\Gamma(\nu)}, \qquad \nu, x > 0$$

denote the density function of the gamma distribution with ν degrees of freedom. The distribution function is given by

$$G_\nu(x) = \int_{-\infty}^{x} g_\nu(y)\,dy$$

$$= \frac{x^\nu e^{-x}}{\Gamma(\nu+1)}\Phi(1,\nu+1;\ x)$$

$$= \frac{x^\nu}{\Gamma(\nu+1)}\Phi(\nu,\nu+1;\ -x). \qquad (1.5)$$

Let X_i be a random variable distributed according to a gamma distribution with ν_i degrees of freedom, i=1, ..., K. Let $X_1, \ldots, X_K$ be jointly independent, and let $X^* = \max(X_1, \ldots, X_K)$. The random variable X^* arises in various statistical problems, such as life testing. The mth moment of X^* is given by

$$\mu_m = E(X^*)^m$$

$$= \sum_{i=1}^{K} \int_0^\infty x^m \prod_{j\neq i}^{m} G_{\nu_j}(x)\, g_{\nu_i}(x)\, dx$$

$$= \sum_{i=1}^{K} \int_0^\infty x^m \prod_{j\neq i}^{m} \frac{x^{\nu_j}}{\Gamma(\nu_j+1)} \Phi(\nu_j, \nu_j+1;-x) g_{\nu_i}(x)dx$$

Using Theorem 1.1 we obtain bounds on the value of μ_m, given by

$$\sum_{i=1}^{K} \int_0^\infty x^m \prod_{j\neq i}^{m} \frac{x^{\nu_j}}{\Gamma(\nu_j+1)} S_{2n-1}(\nu_j,\nu_j+1;-x)g_{\nu_i}(x)dx \leq \mu_m \leq$$

$$\sum_{i=1}^{K} \int_0^\infty x^m \prod_{j\neq i}^{m} \frac{x^{\nu_j}}{\Gamma(\nu_j+1)} S_{2n-2}(\nu_j,\nu_j+1;-x)g_{\nu_i}(x)dx \qquad (1.6)$$

where n is any positive integer and

$$S_n(a,b;x) = \sum_{r=0}^{n} \frac{(a)_r}{(b)_r} \cdot \frac{x^r}{r!}$$

denotes the partial sum of the confluent hypergeometric series $\Phi(a,b;x)$. The left side of the inequality (1.6) reduces to

$$\left(\prod_{j=1}^{K} \Gamma(\nu_j)\right)^{-1} \sum{}^* (-1)^{\sum_{j\neq i} s_j} \Gamma(m+\sum_{j=1}^{K}\nu_j+\sum_{j\neq i}s_j)\prod_{j\neq i}^{m} ((\nu_j+s_j)s_j!)^{-1}$$

where $\sum^*$ denotes summation over all non-negative integer values of $s_j \leq 2n-1$, $j\neq i$. The right hand side of (1.6) is reduced similarly. The relation (1.6) is useful in the computation of μ_m.

For another application consider a non-central gamma distribution with ν degrees of freedom and non-centrality parameter δ, given by the density function

$$g(x) = e^{-\delta} \sum_{r=0}^{\infty} \frac{\delta^r}{r!} g_{\nu+r}(x).$$

The mth moment of the distribution is given by

$$\begin{aligned}
\mu_m(\delta) &= e^{-\delta} \sum_{r=0}^{\infty} \frac{\delta^r}{r!} \int_0^{\infty} x^m g_{\nu+r}(x)\,dx \\
&= e^{-\delta} \sum_{r=0}^{\infty} \frac{\delta^r \Gamma(m+\nu+r)}{\Gamma(\nu+r)\,r!} \\
&= \frac{\Gamma(m+\nu)}{\Gamma(\nu)} e^{-\delta} \Phi(m+\nu,\nu;\delta) \\
&= \frac{\Gamma(m+\nu)}{\Gamma(\nu)} \Phi(-m,\nu;-\delta) \qquad (1.7)
\end{aligned}$$

If $-\nu < m < 0$ the confluent hypergeometric function $\Phi(-m,\nu;-\delta)$ is completely monotone in δ. Then the results of Theorem 1.1 can be used to derive bounds on the value of $\mu_m(\delta)$.

References

[1] Abramowitz, M. and Stegun, I. A. (1970). Handbook of Mathematical Functions. Dover Publications.

[2] Feller, W. (1965). An Introduction to Probability Theory and It's Applications, Vol. II. Wiley Publications.

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